Weyl Group Invariance and p-brane Multiplets

H. Lü[†], C.N. Pope[†]

Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843

K.S. Stelle[⋆]

The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK

ABSTRACT

In this paper, we study the actions of the Weyl groups of the U duality groups for type IIA string theory toroidally compactified to all dimensions $D \geq 3$. We show how these Weyl groups implement permutations of the field strengths, and we discuss the Weyl group multiplets of all supersymmetric p-brane solitons.

1 Introduction

It has been known for a long time that the maximal supergravity theory in D dimensions obtained by Kaluza-Klein dimensional reduction from $D = 11$ has a $G \simeq E_{n(+n)}(\mathbb{R})$ symmetry, where $n = 11 - D$ [1]. Recently it was conjectured that the $G(\mathbb{Z}) \simeq E_{n(+n)}(\mathbb{Z})$ discrete subgroup of this is an exact symmetry of the corresponding string theory, known as U duality [2, 3]. In general, the solutions of a supergravity theory will form infinite-dimensional multiplets under U duality. In [3], this symmetry was used to fill out a full set of 56 purely electric and purely magnetic 4-dimensional black holes that break half of the supersymmetry of $N = 8$ supergravity. Of course, these are only a finite subset of the infinite dimensional $E_{7(+7)}(\mathbb{Z})$ multiplet; the other members of the multiplet have more complicated field-strength configurations involving both electric and magnetic contributions, together with some non-vanishing axions (*i.e.* 0-form potentials for 1-form field strengths). However, since the 56 purely electric and purely magnetic solutions are of primary interest, in view of their interpretation as fundamental quantum states of the string theory, it is useful to isolate a subgroup of $E_{7(+7)}(\mathbb{Z})$ that maps these solutions among themselves. We shall show that the group that does this is the Weyl group of $E_{7(+7)}$. In this paper, we shall study the U Weyl group invariance of all maximal supergravity theories in dimensions $D \geq 3$. We shall also study the corresponding multiplet structures of p-brane solutions under the U Weyl group.

The significance of the Weyl subgroup W of the U duality group G is analogous to that of the $\mathbb{Z}_2$ subgroup of the $U(1)$ electric-magnetic duality group in Maxwell theory, describing the discrete interchange of electric and magnetic fields: $E \longrightarrow B$ and $B \longrightarrow -E$. Thus W is the subgroup of the U duality group that implements certain permutations of the field strengths in the theory, while maintaining their alignment along the axes of the space of field strengths. In other words, the U Weyl group describes certain 90-degree rotations in the space of field strengths. By contrast, the full U duality group G, as well as its quantum restriction $G(\mathbb{Z})$, include intermediate rotations in the space of field strengths. In terms of p-brane solutions, therefore, the U Weyl group preserves the total number of electric and magnetic charges, whereas the full U duality group does not. Thus the U Weyl group gives a characterisation of the independent p-brane solutions of a given type.

The U Weyl group W has another interpretation when acting on the space of supergravity solutions. The scalar fields of a supergravity theory admitting flat space as a solution (*i.e.* of an "ungauged" supergravity) do not have potentials, and hence their asymptotic values are unfixed by the equations of motion. These asymptotic values constitute the moduli of a supergravity solution; since certain subsets of the scalars also occur in exponential prefactors of the antisymmetric tensor field kinetic terms, the moduli also determine the coupling constants of the theory in a particular asymptotically-defined "vacuum." Performing a $G(\mathbb{Z})$ U duality transformation on a supergravity

solution will generally change the asymptotic values of the scalars, and hence change the "vacuum." A subgroup of $G(\mathbb{Z})$ exists, however, that does not change the vacuum. This stability duality group of the vacuum coincides precisely with the Weyl group as defined above when the scalars are taken to tend asymptotically to zero, which will be the case for the family of p-brane solutions considered here.

In identifying the subgroup W of a U duality group G that preserves the total number of electric and magnetic charges as the Weyl group of G, we are not using the classic definition [4] of the Weyl group as the quotient of the normalizer divided by the centralizer of the Cartan subalgebra of G. Since this standard definition is a quotient-group construction, it is not even clear in general that its result properly defines a subgroup of G. Nonetheless, the group G may still have discrete subgroups that are isomorphic to W, even though they are not obtained by the classic quotient construction. This will be the case for the discrete subgroups considered in this paper. We shall identify them as Weyl groups by their explicit actions on the vectors of scalar fields ("dilatonic scalars") appearing in the exponential prefactors of antisymmetric tensor field strengths in the action. These dilaton vectors will transform under the action of our discrete W subgroups in exactly the same way as weight vectors of the irreducible representations of G transform under the Weyl groups, thus giving us the identification.

The U duality group $G(\mathbb{Z})$ has several important subgroups, namely the T, S, and X dualities. T duality [5] is a perturbative symmetry of string theory, valid order by order in the string coupling constant g, although non-perturbative in α'. It is a $D_{n-1} \simeq SO(n-1, n-1; \mathbb{Z})$ subgroup of the U duality group $G(\mathbb{Z}) \simeq E_{n(+n)}(\mathbb{Z})$, which acts on the compactified internal dimensions. The T duality group preserves the NS-NS and R-R sectors of the theory. X duality is an $SL(2, \mathbb{Z})$ subgroup of $E_{n(+n)}(\mathbb{Z})$ that interchanges the NS-NS with the R-R fields. It is a conjectured non-perturbative symmetry of the string theory. In the type IIB context, it is already present in $D = 10$ [6]; in type IIA, it is present for $D = 9$ [7] and lower. Finally, S duality [2], which exists only in $D = 4$, is another $SL(2, \mathbb{Z})$ subgroup of $E_{7(+7)}(\mathbb{Z})$, again non-perturbative. One way to see why four is a special dimension in this regard is to note that whereas the T duality group D_{n-1} is a maximal subgroup of E_n for general values of n, the case $n = 7$ is special, since then $D_6 \times A_1$ is a maximal subgroup of E_7. The A_1 factor corresponds precisely to the $SL(2, \mathbb{Z})$ S duality. S duality preserves the NS-NS and R-R sectors; in particular, it rotates between each NS-NS 2-form field strength and its dual [8]. To summarise, both T and S duality preserve the NS-NS and R-R sectors, the former being perturbative whilst the latter is non-perturbative, interchanging strong and weak couplings. X duality is a non-perturbative symmetry, which interchanges the NS-NS and R-R sectors. In this paper, we shall also discuss the Weyl group symmetries of the S, T and X duality groups. As in the

case of the U Weyl group, the S, T and X Weyl groups also describe certain 90-degree rotations in the space of field strengths.

The bosonic sector of the Lagrangian for $D = 11$ supergravity is $\mathcal{L} = \hat{e}\hat{R} - \frac{1}{48}\hat{e}\,\hat{F}_4^2 + \frac{1}{6}\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3$, where $\hat{A}_3$ is the 3-form potential for the 4-form field strength $\hat{F}_4$. Upon Kaluza-Klein dimensional reduction to D dimensions, this yields the following Lagrangian:

$$
\begin{aligned}
\mathcal{L} \;=\; & eR - \tfrac{1}{2}e\,(\partial\vec{\phi})^2 - \tfrac{1}{48}e\,e^{\vec{a}\cdot\vec{\phi}}\,F_4^2 - \tfrac{1}{12}e\sum_i e^{\vec{a}_i\cdot\vec{\phi}}\,(F_3^i)^2 - \tfrac{1}{4}e\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\,(F_2^{ij})^2 \\
& -\tfrac{1}{4}e\sum_i e^{\vec{b}_i\cdot\vec{\phi}}\,(\mathcal{F}_2^i)^2 - \tfrac{1}{2}e\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,(F_1^{ijk})^2 - \tfrac{1}{2}e\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,(\mathcal{F}_1^{ij})^2 + \mathcal{L}_{FFA}\,,
\end{aligned}
\qquad (1.1)
$$

where F_4, F_3^i, F_2^{ij} and F_1^{ijk} are the 4-form, 3-forms, 2-forms and 1-forms coming from the dimensional reduction of $\hat{F}_4$ in $D = 11$; $\mathcal{F}_2^i$ are the 2-forms coming from the dimensional reduction of the vielbein, and $\mathcal{F}_1^{ij}$ are the 1-forms coming from the dimensional reduction of these 2-forms. The quantity $\vec{\phi}$ denotes an $(11 - D)$-component vector of scalar fields, which we refer to as dilatonic scalars, arising from the dimensional reduction of the elfbein. These scalars appear undifferentiated, *via* the exponential prefactors for the antisymmetric-tensor kinetic terms, and they should be distinguished from the remaining spin-0 quantities in D dimensions, namely the axions, *i.e.,* the 0-form potentials A_0^{ijk} and $\mathcal{A}_0^{ij}$. These axion fields have constant shift symmetries, under which the action is invariant, and are thus properly thought of as 0-form potentials rather than true scalars. In particular, their 1-form field strengths can adopt topologically non-trivial configurations, corresponding to electrically-charged or magnetically-charged p-brane solitons.

The term $\mathcal{L}_{FFA}$ comes from the dimensional reduction of the $\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3$ term in $D = 11$. Note that the field strengths appearing in the kinetic terms in (1.1) in general acquire "Chern-Simons" modifications, as a consequence of the dimensional reduction procedure. As we shall see later in detail, both of these additional types of term play only a subsidiary role in the discussion of the Weyl group symmetries.

The remaining aspect of the bosonic Lagrangian (1.1) that we need to address is the set of constant vectors $\vec{a}_{i\cdots j}$ and $\vec{b}_{i\cdots j}$ appearing in the exponential prefactors of the kinetic terms for the various antisymmetric tensors. As was shown in [9], these "dilatonic vectors" can be expressed as follows:

	F_{MNPQ}	Vielbein	
4 $-$ form :	$\vec{a} = -\vec{g}\,,$		
3 $-$ forms :	$\vec{a}_i = \vec{f}_i - \vec{g}\,,$		(1.2)
2 $-$ forms :	$\vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}\,,$	$\vec{b}_i = -\vec{f}_i$	
1 $-$ forms :	$\vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}\,,$	$\vec{b}_{ij} = -\vec{f}_i + \vec{f}_j\,,$	

where the vectors $\vec{g}$ and $\vec{f_i}$ have $(11 - D)$ components in D dimensions, and satisfy

$$\vec{g} \cdot \vec{g} = \frac{2(11-D)}{D-2}, \qquad \vec{g} \cdot \vec{f_i} = \frac{6}{D-2}, \qquad \vec{f_i} \cdot \vec{f_j} = 2\delta_{ij} + \frac{2}{D-2}. \tag{1.3}$$

Note that the definitions in (1.2) are given for $i < j < k$, and that the vectors $\vec{a}_{ij}$ and $\vec{a}_{ijk}$ are antisymmetric in their indices. The 1-forms $\mathcal{F}_{Mi}^{(j)}$ and hence the vectors b_{ij} are only defined for $i < j$, but it is sometimes convenient to regard them as being antisymmetric too, by defining $\vec{b}_{ij} = -\vec{b}_{ji}$ for $i > j$.

Eqns (1.2) and (1.3) contain all the information we shall need about the dilaton vectors in D-dimensional maximal supergravity. The explicit forms of the vectors $\vec{f_i}$ and $\vec{g}$ that result from the dimensional reduction of $D = 11$ supergravity are

$$\begin{aligned}
\vec{g} &= 3(s_1, s_2, \ldots, s_{11-D}), \\
\vec{f_i} &= \Big(\underbrace{0, 0, \ldots, 0}_{i-1}, (10-i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{11-D} \Big),
\end{aligned} \tag{1.4}$$

where $s_i = \sqrt{2/((10-i)(9-i))}$. Of course, the explicit forms of these vectors are inessential because they depend on the specific dimensional-reduction procedure used. For example, one can also obtain lower dimensional supergravities by dimensional reduction of type IIB supergravity in $D = 10$. In this case, the explicit forms of the vectors $\vec{f_i}$ and $\vec{g}$ are different from the ones given in (1.4). However, these forms are related to those of (1.4) by an orthogonal transformation of the dilaton vectors involving a rotation of the first two components, ϕ_1 and ϕ_2; hence they also satisfy the dot product relations (1.3). Specifically, in the type IIB basis, the vectors are given by

$$\begin{aligned}
\vec{g} &= 3(0, -\frac{2}{3\sqrt{7}}, , s_3, s_4, \ldots, s_{11-D}), \\
\vec{f_1} &= (1, -\frac{3}{\sqrt{7}}, s_3, s_4, \ldots, s_{11-D}), \\
\vec{f_2} &= (-1, -\frac{3}{\sqrt{7}}, s_s, s_4, \ldots, s_{11-D}),
\end{aligned} \tag{1.5}$$

with $\vec{f_i}$ for $i \geq 3$ the same as in (1.4).

Having obtained the full bosonic Lagrangian for D-dimensional maximal supergravity, we are in a position to discuss its symmetry under the Weyl group of the supergravity symmetry group. In section 2, we shall examine the $E_{2(+2)}(\mathbb{R}) \simeq GL(2, \mathbb{R})$ symmetry group of $D = 9$ supergravity in detail, and show how the discrete $\mathbb{Z}_2$ Weyl group emerges as the group of permutations of pairs of field-strength tensors. In addition, we shall show that this $\mathbb{Z}_2$ Weyl group symmetry exists also in type IIB supergravity in $D = 10$. In section 3, we generalise the discussion to maximal supergravities in $3 \leq D \leq 8$ dimensions. In section 4, we shall discuss the Weyl groups of S, T and

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X dualities, which are subgroups of the U duality. Section 5 contains an analysis of the multiplet structure of p-brane solutions under the Weyl-group symmetries. We shall also obtain a new supersymmetric p-brane solution in $D = 3$, which involves 8 participating 1-form field strengths. The paper ends with discussion and conclusions in section 6.

2   The U Weyl group in $D \geq 9$

We begin this section by discussing the symmetries of $D = 9$ supergravity. This theory can be obtained by dimensional reduction of $D = 10$ type IIA supergravity, which in turn can be obtained by reduction from $D = 11$ supergravity. Alternatively, the same $D = 9$ supergravity can be obtained from the dimensional reduction of type IIB supergravity in $D = 10$. The two reduction routes result in two formulations of the $D = 9$ theory that are related by an orthogonal field redefinition of the dilatonic scalars. We shall construct $D = 9$ supergravity here by dimensional reduction of type IIA supergravity. However, for convenience, we shall then choose the basis of dilatonic scalar fields that corresponds to the type IIB reduction route. The bosonic sector of the theory contains the vielbein, a dilaton ϕ together with a second dilatonic scalar φ, one 4-form field strength F_4, two 3-forms F_3^i, three 2-forms F_2^{12} and $\mathcal{F}_2^i$ and one 1-form $\mathcal{F}_1^{12} = d\chi$. It follows from (1.1), (1.2) and (1.5) that the bosonic Lagrangian is given by

$$
\begin{aligned}
\mathcal{L} =\ & eR - \tfrac{1}{2}e(\partial\phi)^2 - \tfrac{1}{2}e(\partial\varphi)^2 - \tfrac{1}{2}ee^{-2\phi}(\partial\chi)^2 \\
& -\tfrac{1}{48}ee^{\frac{2}{\sqrt{7}}\varphi}(F_4)^2 - \tfrac{1}{2}ee^{\phi-\frac{1}{\sqrt{7}}\varphi}(F_3^{(1)})^2 - \tfrac{1}{2}ee^{-\phi-\frac{1}{\sqrt{7}}\varphi}(F_3^{(2)})^2 - \tfrac{1}{4}ee^{-\frac{4}{\sqrt{7}}\varphi}(F_2^{(12)})^2 \qquad (2.1) \\
& -\tfrac{1}{4}ee^{-\phi+\frac{3}{\sqrt{7}}\varphi}(\mathcal{F}_2^{(1)})^2 - \tfrac{1}{4}ee^{\phi+\frac{3}{\sqrt{7}}\varphi}(\mathcal{F}_2^{(2)})^2 - \tfrac{1}{2}\tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{(12)} - \tilde{F}_3^{(1)} \wedge \tilde{F}_3^{(2)} \wedge A_3 \ ,
\end{aligned}
$$

where we have defined $\phi = \phi_1$, $\varphi = \phi_2$, and $\chi = \mathcal{A}_0^{(12)}$. We are using the notation that field strengths without tildes include the various Chern-Simons modifications, whilst field strengths written with tildes do not include the modifications. Thus we have:

$$
\begin{aligned}
F_4 &= \tilde{F}_4 - \tilde{F}_3^{(1)} \wedge \mathcal{A}_1^{(1)} - \tilde{F}_3^{(2)} \wedge \mathcal{A}_1^{(2)} - \tfrac{1}{2}\tilde{F}_2^{(12)} \wedge \mathcal{A}_1^{(1)} \wedge \mathcal{A}_1^{(2)} \ , \\
F_3^{(1)} &= \tilde{F}_3^{(1)} - \tilde{F}_2^{(12)} \wedge \mathcal{A}_1^{(2)} \ , \\
F_3^{(2)} &= \tilde{F}_3^{(2)} + F_2^{(12)} \wedge \mathcal{A}_1^{(1)} - \chi(\tilde{F}_3^{(1)} - F_2^{(12)} \wedge \mathcal{A}_1^{(2)}) \ , \\
F_2^{(12)} &= \tilde{F}_2^{(12)} \ , \qquad \mathcal{F}_2^{(1)} = \mathcal{F}_2^{(1)} + \chi\mathcal{F}_1^{(2)} \ , \qquad \mathcal{F}_2^{(2)} = \tilde{\mathcal{F}}_2^{(2)} \ , \qquad \mathcal{F}_1^{(12)} = \tilde{\mathcal{F}}_1^{(12)} \ . \qquad (2.2)
\end{aligned}
$$

In obtaining these expressions from the results in [9], we have performed the field redefinition $\mathcal{A}_1^{(1)} \longrightarrow \mathcal{A}_1^{(1)} - \chi\,\mathcal{A}_1^{(2)}$ in order to arrive at a formulation where, as we shall see below, $SL(2,\mathbb{R})$ acts linearly on the potentials $\mathcal{A}_1^i$.

The Lagrangian (2.1) has a $GL(2, \mathbb{R}) \simeq SL(2, \mathbb{R}) \times SO(1, 1; \mathbb{R})$ symmetry [1]. The $SO(1, 1; \mathbb{R})$ symmetry corresponds to a constant shift of the dilatonic scalar φ, together with rescalings of the gauge potentials:

$$\varphi \longrightarrow \varphi + 2\sqrt{7}c \ ,$$

$$A_3 \longrightarrow e^{-2c} A_3 \ , \qquad A_1^{(12)} \longrightarrow e^{4c} A_1^{(12)} \ ,$$

$$A_2^{(i)} \longrightarrow e^{c} A_2^{(i)} \ , \qquad \mathcal{A}_1^{(i)} \longrightarrow e^{-3c} \mathcal{A}_1^{(i)} \ . \tag{2.3}$$

In order to understand the action of the $SL(2, \mathbb{R})$ symmetry, it is convenient to re-express the scalar fields ϕ and χ, which live in the coset $SL(2, \mathbb{R})/SO(2)$, in terms of three fields (X, Y, Z) subject to the $SO(1, 2)$-invariant constraint $X^2 - Y^2 - Z^2 = 1$:

$$
\begin{aligned}
X &= \cosh\phi + \tfrac{1}{2}\chi^2 e^{-\phi} \ , \\
Y &= \sinh\phi + \tfrac{1}{2}\chi^2 e^{-\phi} \ , \\
Z &= \chi e^{-\phi} \ .
\end{aligned}
\tag{2.4}
$$

The scalar Lagrangian for the fields ϕ and χ now takes the simple $SO(1, 2)$-invariant form

$$-\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{-2\phi}(\partial\chi)^2 = \tfrac{1}{2}(\partial X)^2 - \tfrac{1}{2}(\partial Y)^2 - \tfrac{1}{2}(\partial Z)^2 \ . \tag{2.5}$$

The group $SO(1, 2)$ is isomorphic to $SL(2, \mathbb{R})$, whose action on (X, Y, Z) is

$$
\begin{pmatrix} Z & Y - X \\ Y + X & -Z \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} Z & Y - X \\ Y + X & -Z \end{pmatrix} \Lambda^{-1} \ , \tag{2.6}
$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \tag{2.7}$$

and $ad - bc = 1$. This implies that

$$e^{-\phi} \longrightarrow (a + b\chi)^2 e^{-\phi} + b^2 e^{\phi} \ , \qquad \chi e^{-\phi} \longrightarrow (a + b\chi)(c + d\chi)e^{-\phi} + bd \, e^{\phi} \ , \tag{2.8}$$

which translates into the fractional linear transformation $\lambda \equiv \chi + i e^{\phi} \to (c + d\lambda)/(a + b\lambda)$. It is straightforward to see that the entire Lagrangian (2.1) is invariant under this $SL(2, \mathbb{R})$ transformation if the potentials transform as

$$A_3 \longrightarrow A_3 \ , \qquad A_1^{(12)} \longrightarrow A_1^{(12)} \ ,$$

$$
\begin{pmatrix} A_2^{(1)} \\ A_2^{(2)} \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} A_2^{(1)} \\ A_2^{(2)} \end{pmatrix} \ , \qquad \begin{pmatrix} \mathcal{A}_1^{(1)} \\ \mathcal{A}_1^{(2)} \end{pmatrix} \longrightarrow (\Lambda^T)^{-1} \begin{pmatrix} \mathcal{A}_1^{(1)} \\ \mathcal{A}_1^{(2)} \end{pmatrix} \ . \tag{2.9}
$$

This $SL(2, \mathbb{R})$ symmetry of $D = 9$ supergravity can also be obtained by dimensionally reducing type IIB supergravity [7].

We may now consider a discrete S_2 subgroup of $SL(2, \mathbb{R})$, corresponding to taking $a = d = 0$, $b = 1$ and $c = -1$. Under this permutation group, the non-invariant fields transform as follows:

$$e^{-\phi} \longrightarrow e^{\phi} + \chi^2 e^{-\phi} , \qquad \chi e^{-\phi} \longrightarrow -\chi e^{-\phi} ,$$

$$A_2^{(1)} \longrightarrow A_2^{(2)} , \qquad A_2^{(2)} \longrightarrow -A_2^{(1)} , \qquad \mathcal{A}_1^{(1)} \longrightarrow \mathcal{A}_1^{(2)} , \qquad \mathcal{A}_1^{(2)} \longrightarrow -\mathcal{A}_1^{(1)} . \qquad (2.10)$$

Note that this transformation simply interchanges the indices 1 and 2 of the pair of 3-form field strengths $\tilde{F}_3^{(i)}$, and the pair of 2-form field strengths $\tilde{\mathcal{F}}_2^{(i)}$, together with certain sign changes. Its action on the untilded field strengths with their Chern-Simons modifications, which appear in the kinetic terms in (2.1), is more complicated. These complications are associated exclusively with terms involving the 0-form potential χ. If we concentrate only on the terms that are independent of the bare undifferentiated χ, which can be thought of as the leading-order terms, we see that the action of this S_2 discrete group is simply to interchange the pairs of field strengths and at the same time interchange their associated dilaton prefactors. This interchange of dilaton vectors is equivalent to $\phi \longrightarrow -\phi$. As far as the other field strengths in the theory are concerned, one can check that the dilaton vectors for the field strengths F_4 and $F_2^{(12)}$ are invariant, since they are independent of ϕ. On the other hand, the S_2 transformation reverses the sign of the dilaton vector for the 1-form field strength $\mathcal{F}_1^{(12)} = d\chi$. The Lagrangian is nevertheless invariant, as we already demonstrated, since, unlike the higher-degree potentials which transform linearly, the 0-form potential undergoes the non-linear transformation $\chi \longrightarrow -\chi e^{-2\phi} + \cdots$, which precisely compensates for the sign reversal of its dilaton vector.

The discrete S_2 subgroup (2.10) is also seen to coincide with the subgroup of the $SL(2, \mathbb{Z})$ U duality group that leaves unchanged the scalar asymptotic values $(\langle \phi \rangle, \langle \chi \rangle)$ when these are taken to vanish. To see this, we observe from (2.8) that to preserve the vanishing asymptotic values $\langle \phi \rangle = 0 = \langle \xi \rangle$, we must have $a^2 + b^2 = 1$ and $ac + bd = 0$, in addition to $ad - bc = 1$. Up to a trivial sign change of the matrix Λ (2.7), the only solutions are the identity and $a = d = 0$, $b = -c = 1$. These are precisely the elements of the S_2 group that we discussed above. This observation gives rise to the alternative interpretation of this S_2 subgroup as the stability duality group of the vacuum, as presaged in the introduction.

As we shall see, the pattern that we have described here occurs also in lower dimensions. In D-dimensional maximal supergravity, the analogous discrete subgroup of the supergravity symmetry group $G \simeq E_{n(+n)}$ ($n = 11 - D$) implements certain permutations of the dilaton vectors associated with the field strengths. Although the 9-dimensional example is rather too degenerate to illustrate

the point clearly, we shall see more transparently in lower dimensions, where the group is larger, that this interchange of the dilaton prefactors corresponds precisely to the action of the Weyl group W of $E_{n(+n)}(\mathbb{R})$. The interpretation of the Weyl group as the stability duality group of the vacuum will also obtain in the lower-dimensional cases.

Now let us consider maximal supergravities in higher dimensions. The symmetry groups of 11-dimensional supergravity and type IIA supergravity are rather trivial, with no non-trivial Weyl group. On the other hand, type IIB supergravity has an $SL(2, \mathbb{R})$ symmetry [6], giving rise to a discrete S_2 Weyl-group symmetry. The analysis of this Weyl group action is analogous to that of $D = 9$ supergravity. Since the self-dual 5-form field strength in type IIB supergravity is inert under $SL(2, \mathbb{R})$, we may omit it from the discussion and consider only the subset of type IIB fields that are subject to $SL(2, \mathbb{R})$ transformations. This subset of fields can be described by a Lagrangian, and in fact since the Lagrangian (2.1) for the bosonic sector of $D = 9$ supergravity is written in the type IIB basis, we may simply obtain the relevant part of the type IIB Lagrangian by setting φ and the 3-form and 1-form potentials to zero in (2.1), and by restoring the dependence on the tenth coordinate. The resulting Lagrangian is therefore invariant under the $SL(2, \mathbb{R})$ transformations given by (2.8) and (2.9). The discrete Weyl-group transformation (2.10) corresponds also to the interchange of the dilaton vectors of the two 3-form field strengths.

It is of interest to note that in addition to the $SL(2, \mathbb{R})$ symmetry in type IIB supergravity, there is a further discrete S_2 symmetry that differs from the Weyl-group symmetry discussed above. Namely, type IIB supergravity is invariant under the following discrete transformation:

$$A_2^{(1)} \longrightarrow -A_2^{(1)} , \qquad A_4 \longrightarrow -A_4 , \qquad \chi \longrightarrow -\chi \qquad (2.11)$$

with the rest of the fields being invariant. This is not part of the $SL(2, \mathbb{R})$ symmetry. Such additional discrete symmetries lying outside the $G \simeq E_{n(+n)}$ symmetry group also arise in lower dimensions.

So far, we have considered transformations that generate the *symmetries* of the supergravity theories. In $D = 9$ dimensions, there is another important transformation, namely the $\mathbb{Z}_2$ T-duality of string theory, which is not a symmetry. Instead, it relates the type IIA string compactified on a large circle to the type IIB string compactified on a small circle. At the level of $D = 9$ maximal supergravity, this is a transformation that maps between the Lagrangians obtained by dimensional reduction of the type IIA and type IIB theories. The T-duality transformation corresponds to making the following field redefinition of the two dilatonic scalars (ϕ, φ):

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{IIA} = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{7}}{4} \\ -\frac{\sqrt{7}}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{IIB} . \qquad (2.12)$$

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This transformation can be derived from the T-duality transformation in the $D = 10$ string σ model. The σ-model transformation that corresponds to the transformation (2.12) of the dilatonic scalar fields is given by [10]

$$\tilde{g}_{zz}^{\sigma} = \frac{1}{g_{zz}^{\sigma}} \, , \qquad \tilde{\phi} = \phi - \tfrac{1}{2} \log g_{zz}^{\sigma} \, , \tag{2.13}$$

where g_{zz}^{σ} denotes the diagonal internal component of the σ model metric g_{MN}^{σ}, upon reduction to $D = 9$. Translating to the Einstein-frame metric $g_{MN} = e^{\frac{1}{2}\phi} g_{MN}^{\sigma}$ in $D = 10$, and performing the dimensional reduction, which implies $ds_{10}^2 = e^{\varphi/(2\sqrt{7})} ds_9^2 + e^{-\sqrt{7}\varphi/2} (dz + \mathcal{A})^2$, we obtain precisely the transformation (2.12). One can easily verify that acting with this transformation on the Lagrangian (2.1) gives rise to the Lagrangian obtained from dimensional reduction of type IIA supergravity, whose dilaton vectors are given by (1.2) and (1.4). Note that if one truncates $D = 9$ maximal supergravity to $N = 1$ supergravity, where only the NS-NS fields remain, then this T-duality is actually a symmetry of the theory.

3   U Weyl group in $3 \le D \le 8$

We saw in the previous section that there is a discrete S_2 symmetry of the $D = 9$ maximal supergravity theory, which has the interpretation of being the Weyl group of the $SL(2, \mathbb{R})$ symmetry of the theory. In this section, we shall extend the discussion to the maximal supergravities in all dimensions $3 \le D \le 8$. We shall see that the discrete Weyl-group symmetry of the theory becomes more apparent as the symmetry group enlarges with the descent through the dimensions.

3.1 $D = 8$

Maximal supergravity in $D = 8$ contains one 4-form field strength F_4; three 3-forms F_3^i; six 2-forms comprising three F_2^{ij} and three $\mathcal{F}_2^{ij}$; and four 1-forms comprising one F_1^{ijk} and three $\mathcal{F}_1^{ij}$, where the indices run over $1, 2, 3$. The dilaton vectors for these field strengths are given by (1.2). To discuss the Weyl group symmetry, we first consider the 4-form and 3-forms. Since the degree of the 4-form field strength is equal to $D/2$, it follows that there is an S_2 symmetry of the equations of motion, corresponding to a dualisation of F_4, under which its dilaton vector $\vec{a}$ reverses its sign. Under this S_2 symmetry, the dilaton vectors of the 3-forms are inert. For the 3-forms, there is however an S_3 symmetry, corresponding to permutations of their dilaton vectors $\vec{a}_i$, under which the dilaton vector for the 4-form is invariant. This independence of the S_2 and S_3 symmetries is not immediately manifest. It can be seen, however, from the detailed properties of the dilaton vectors.

In particular, it follows from (1.2) and (1.3) that in $D = 8$ these dilaton vectors satisfy

$$\vec{a} \cdot \vec{a}_i = 0 \ , \qquad \sum_i \vec{a}_i = 0 \ . \tag{3.1}$$

Thus the vectors $\vec{a}_i$ form a plane, on which the S_3 symmetry acts, whilst the S_2 reflects the vectors orthogonal to this plane. This $S_3 \times S_2$ discrete symmetry is isomorphic to the Weyl group of $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, which is the symmetry group of $D = 8$ maximal supergravity [1]. In fact, the vectors $\frac{1}{\sqrt{2}} \vec{a}_i$ and $\pm \frac{1}{\sqrt{2}} \vec{a}$ are precisely the weight vectors of the fundamental representations of $SL(3, \mathbb{R})$ and $SL(2, \mathbb{R})$ respectively. This can be seen by noting that the weight vectors $\vec{h}_i$ of the fundamental representation of $SL(N, \mathbb{R})$ satisfy

$$\vec{h}_i \cdot \vec{h}_j = \delta_{ij} - \frac{1}{N} \ , \qquad \sum_i \vec{h}_i = 0 \ , \tag{3.2}$$

which is precisely the relation satisfied by $\pm \frac{1}{\sqrt{2}} \vec{a}$ and $\frac{1}{\sqrt{2}} \vec{a}_i$, for $N = 2$ and $N = 3$ respectively. These weight vectors permute among themselves under the actions of the corresponding S_3 and S_2 Weyl groups.

To complete the discussion of the U Weyl group symmetry in $D = 8$ supergravity theory, we need to examine its action on the dilaton vectors of the 2-form and 1-form field strengths. These dilaton vectors can be expressed in terms of $\vec{a}_i$ and $\vec{a}$ as follows:

$$\begin{aligned} 2 - \text{forms}: && \vec{a}_{ij} = -\vec{a}_k - \vec{a} \ , && \vec{b}_i = -\vec{a}_i + \vec{a} \ , \\ 1 - \text{forms}: && \vec{a}_{123} = 2\vec{a} \ , && \vec{b}_{ij} = -\vec{a}_i + \vec{a}_j \ , \end{aligned} \tag{3.3}$$

where in the first line, k takes the value in the set $\{1, 2, 3\}$ that is not equal to either i or j. Thus under the S_3 group, the two sets of three 2-form dilaton vectors a_{ij} and b_i permute among themselves. Under the S_2 group, the two sets interchange. For the 1-forms, we see that the dilaton vector $\vec{a}_{123}$ is inert under S_3, but has a sign change under S_2. The dilaton vectors b_{ij} permute with sign changes under the S_3, and are inert under the S_2. As we saw in the $D = 9$ example in the previous section, the 0-form potentials, unlike the higher-rank potentials, transform non-linearly, in a manner that precisely compensates for the sign changes of their dilaton vectors. Thus, we see that the six dilaton vectors of the 2-form field strengths form an irreducible $(3, 2)$ representation of the $S_3 \times S_2$ Weyl group of $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$. The dilaton vectors $\vec{a}_{123}$ and $\vec{b}_{ij}$ for the 1-form field strengths, together with their negatives, transform as the $(1, 2)$ and $(6, 1)$ representations.

Having observed that the dilaton vectors of the various field strengths form representations of the Weyl group of $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, we now argue that the Weyl group is indeed a discrete symmetry of the theory. First, note that if we permute the field strengths appearing in the Lagrangian (1.1) at the same time as their dilaton vectors are permuted according to the rule given above, the form

of the Lagrangian is preserved. Of course, this of itself does not demonstrate the invariance of the theory under the Weyl-group action, since, as we already saw for $D = 9$ in the previous section, the Chern-Simons modifications to the field strengths complicate the discussion considerably. However, as we also saw in the previous section, the Chern-Simons modifications are sub-leading terms that follow in a prescribed fashion once the leading-order behaviour of the unmodified field strengths is established. The details of how these sub-leading terms conspire to make the theory fully invariant are contained in the standard proof of the $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ invariance of $D = 8$ supergravity [1]. Thus we may reduce the discussion of the Weyl-group invariance to a discussion of the leading-order terms by invoking these known results, provided we can show that the Weyl group, as identified in this paper, is contained within the full supergravity symmetry group $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$.

The S_2 symmetry reversing the sign of the dilaton vector for the 4-form field strength corresponds to the symmetry of interchanging the equations of motion and the Bianchi identity of the 4-form field strength and its dual. This is part of the $SL(2, \mathbb{R})$ symmetry. Under $SL(3, \mathbb{R})$, the 2-form potentials A_2^i transform as $A_2^i \longrightarrow \Lambda^i{}_j A_2^j$, where $\Lambda^i{}_j$ is an $SL(3, \mathbb{R})$ matrix. It is easy to see that there is a discrete subgroup of these matrices that permutes the A_2^i's, with certain sign changes as necessary in order to ensure that the matrices have determinant 1. This permutation of the 2-form potentials corresponds precisely to the previously-discussed S_3 permutation of the dilaton vectors of the 3-form field strengths. The sign changes of the potentials are unimportant, since the leading-order terms in the Lagrangian are quadratic in potentials. This completes the demonstration of the invariance of $D = 8$ maximal supergravity under the $S_3 \times S_2$ U Weyl group.

3.2 $D = 7$

There are a total of five 3-form field strengths in $D = 7$ maximal supergravity: four F_3^i and one $F_3 = {}^* F_4$, which is the dualisation of the 4-form field strength. The index i runs over $1, 2, 3, 4$. The dilaton vectors $\vec{a}_i$ of the the field strengths F_3^i are given by (1.2). The dilaton vector of F_3, which we denote by $\vec{a}_5$, is given by $\vec{a}_5 = -\vec{a}$. It follows from (1.2) and (1.3) that these vectors satisfy

$$\tfrac{1}{\sqrt{2}} \vec{a}_I \cdot \tfrac{1}{\sqrt{2}} \vec{a}_J = \delta_{IJ} - \tfrac{1}{5} , \qquad \sum_I \vec{a}_I = 0 , \tag{3.4}$$

where $I = (i, 5)$. These are precisely the defining properties of the weight vectors of the 5-dimensional fundamental representation of $SL(5, \mathbb{R})$. Thus these dilaton vectors are permuted and form a 5-component irreducible multiplet under the action of the Weyl group S_5 of $SL(5, \mathbb{R})$.

Now let us examine how the dilaton vectors of the lower-rank field strengths transform under this Weyl group. There are a total of ten 2-form field strengths: six corresponding to the dilaton vectors $\vec{a}_{ij}$ and four corresponding to $\vec{b}_i$. There are ten 1-form field strengths: four corresponding

to $\vec{a}_{ijk}$ and six corresponding to $\vec{b}_{ij}$. From (1.2) and (3.4), we see that they can be expressed in terms of $\vec{a}_i$ and $\vec{a}_5$ as

$$2 - \text{forms}: \qquad \vec{a}_{ij} = \vec{a}_i + \vec{a}_j + \vec{a}_5 , \qquad \vec{b}_i = -\vec{a}_i - \vec{a}_5 ,$$
$$1 - \text{forms}: \qquad \vec{a}_{ijk} = -\vec{a}_\ell + \vec{a}_5 , \qquad \vec{b}_{ij} = -\vec{a}_i + \vec{a}_j , \tag{3.5}$$

where in the second line, ℓ takes the value in the set $\{1, 2, 3, 4\}$ that is different from i, j and k. Owing to the property that $\sum_I \vec{a}_I = 0$, it is easy to see that under the S_5 permutations of $\vec{a}_I$, the dilaton vectors of the 2-form field strengths form a 10-component irreducible multiplet. The ten dilaton vectors of the 1-form field strengths, together with their negatives, form a 20-component multiplet. In fact, up to an overall rescaling by $\sqrt{2}$, they are precisely the non-zero weights of the adjoint representation of $SL(5, \mathbb{R})$.

To leading order, the Lagrangian will be invariant under the S_5 Weyl group if we permute the associated field strengths at the same time as their dilaton vectors. As in the previous discussions for $D = 9$ and $D = 8$, the complete proof of invariance, including the Chern-Simons modifications, follows by invoking the known $SL(5, \mathbb{R})$ invariance of $D = 7$ supergravity. Noting that the 2-form potentials A_2^I transform under $SL(5, \mathbb{R})$ as $A_2^{I'} = \Lambda^I{}_J A_2^J$, we see that there is a discrete subgroup S_5 of matrices that permutes the A_2^i's, with certain sign changes. This precisely corresponds to the permutation of the associated field strengths in the leading-order terms of the Lagrangian.

3.3 $D = 6$

In $D = 6$, we can dualise F_4 to a 2-form field strength, and then the highest rank of the field strengths is 3. There are five 3-form field strengths, with corresponding dilaton vectors $\vec{a}_i$ given in (1.2). Since the 3-form field strengths have rank $D/2$, there are additional symmetries at the level of the equations of motion, which interchange the field equations and the Bianchi identities for the field strengths and their duals. Such duality transformations are associated with sign changes of the dilaton vectors $\vec{a}_i$. Thus for the 3-form field strengths considered in isolation, there is a symmetry consisting of permutations of the $\vec{a}_i$ together with an arbitrary number of sign changes.

Now we examine how this symmetry acts on the dilaton vectors for the 2-form and 1-form field strengths. There are a total of sixteen 2-form field strengths, corresponding to dilaton vectors $\vec{a}_{ij}$, $\vec{b}_i$ and $-\vec{a}$, and there are twenty 1-form field strengths, corresponding to dilaton vectors $\vec{a}_{ijk}$ and $\vec{b}_{ij}$. The vectors are given in terms of $\vec{a}_i$ by

$$2 - \text{forms}: \quad \vec{a}_{ij} = \vec{a}_i + \vec{a}_j - \tfrac{1}{2}\sum_k \vec{a}_k , \qquad \vec{b}_i = -\vec{a}_i + \tfrac{1}{2}\sum_k \vec{a}_k , \qquad -\vec{a} = -\tfrac{1}{2}\sum_k \vec{a}_k ,$$
$$1 - \text{forms}: \quad \vec{a}_{ijk} = \vec{a}_i + \vec{a}_j + \vec{a}_k - \sum_\ell \vec{a}_\ell , \qquad \vec{b}_{ij} = -\vec{a}_i + \vec{a}_j . \tag{3.6}$$

We may now observe that the full group of permutations and sign changes of the vectors $\vec{a}_i$ that we have discussed above does not map the set of 2-form dilaton vectors into itself. However, the set does map into itself if we impose the restriction that the sign changes must occur in *pairs*. This is precisely the action of the Weyl group of $D_5 \simeq SO(5,5)$ on the weight vectors of its 10-dimensional fundamental representation. In fact, the ten vectors $\pm \frac{1}{\sqrt{2}}\vec{a}_i$ are precisely the weight vectors of this representation, since $\vec{a}_i \cdot \vec{a}_j = 2\delta_{ij}$. The dilatons for the 2-form field strengths form a 16-component irreducible multiplet of the Weyl group of $SO(5,5)$. The 20 dilaton vectors of the 1-form field strengths, together with their negatives, form a 40-component multiplet of the Weyl group, and they are in fact $\sqrt{2}$ times the non-zero weights of the adjoint representation of $SO(5,5)$.

Following the same logic as used in the cases $D \geq 7$, we can show that this U Weyl group action is a symmetry of the full $D = 6$ supergravity theory.

3.4 $D = 5, 4, 3$

In $D = 5$, the 4-form field strength is dualised to a 1-form, and the six 3-forms are dualised to 2-forms. Thus we have a total of twenty seven 2-forms, corresponding to 15 dilaton vectors $\vec{a}_{ij}$, 6 vectors $\vec{b}_i$ and 6 vectors $a_i^* = -\vec{a}_i$, where the $*$ denotes the dilaton vectors of the dualised 3-forms. These 27 dilaton vectors are given by (1.2) and (1.3), and we find that in $D = 5$, the vectors $\vec{f}_i$ and $\vec{g}$ can be expressed as

$$\vec{f}_i = \sqrt{2}\vec{e}_i + (\tfrac{1}{\sqrt{6}} - \tfrac{1}{3\sqrt{2}})\sum_j \vec{e}_j \ , \qquad \vec{g} = \sqrt{\tfrac{2}{3}}\sum_j \vec{e}_j \ , \tag{3.7}$$

where $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. It turns out that the 27 dilaton vectors, divided by $\sqrt{2}$, are precisely the weight vectors of the 27-dimensional fundamental representation of E_6. To see this, we note that the simple roots of E_6 are given by

$$\vec{\alpha}_1 = \vec{e}_2 - \vec{e}_3 \ , \qquad \vec{\alpha}_2 = \vec{e}_3 - \vec{e}_4 \ , \qquad \vec{\alpha}_3 = \vec{e}_4 - \vec{e}_5 \ , \qquad \vec{\alpha}_4 = \vec{e}_4 + \vec{e}_5 \ ,$$

$$\vec{\alpha}_5 = \tfrac{1}{2}(\vec{e}_1 - \vec{e}_2 - \vec{e}_3 - \vec{e}_4 - \vec{e}_5) + \tfrac{\sqrt{3}}{2}\vec{e}_6 \ , \qquad \vec{\alpha}_5 = \tfrac{1}{2}(\vec{e}_1 - \vec{e}_2 - \vec{e}_3 - \vec{e}_4 + \vec{e}_5) - \tfrac{\sqrt{3}}{2}\vec{e}_6 \ . \tag{3.8}$$

The action of the Weyl group of E_6 can be generated by the six Weyl reflections S_i in the hyperplanes orthogonal to the simple roots $\vec{\alpha}_i$, whose actions on any vector $\vec{\gamma}$ are given by $S_i(\vec{\gamma}) = \vec{\gamma} - (\vec{\gamma} \cdot \vec{\alpha}_i)\vec{\alpha}_i$. The highest-weight vector $\vec{\mu}$ of the 27-dimensional fundamental representation is defined by $\vec{\alpha}_i \cdot \vec{\mu} = \delta_{i6}$. The rest of the 26 weight vectors can be obtained by acting with the Weyl group on the highest weight vector. We find that the 27 dilaton vectors for the 2-form field strengths are $\sqrt{2}$ times the weight vectors of the 27-dimensional representation of E_6, after an orthogonal transformation of the basis for the unit vectors $\vec{e}_i$.

There are thirty six 1-form field strengths, corresponding to 20 dilaton vectors $\vec{a}_{ijk}$, 15 vectors $\vec{b}_{ij}$ and one vector $\vec{a}^* = -\vec{a}$. These vectors, together with their negatives, form a 72-component multiplet of the Weyl group of E_6. It is easy to verify that they correspond to the non-zero weights of the adjoint representation of E_6.

Now let us consider $D = 4$ maximal supergravity. The seven 3-form field strengths F_3^i are dualised to 1-form field strengths. Thus there are twenty eight 2-form field strengths, corresponding to $\vec{a}_{ij}$ and $\vec{b}_i$, and there are sixty three 1-form field strengths, corresponding to $\vec{a}_{ijk}$, $\vec{b}_{ij}$ and $\vec{a}_i^* = -\vec{a}_i$. Since the rank of the 2-form field strengths is $D/2$, there is a duality symmetry under which their equations of motion and Bianchi identities are interchanged. This leads to an enlarged symmetry group under which the 2-form dilaton vectors and their negatives are treated on an equal footing. We find that the 56 vectors $\pm\frac{1}{\sqrt{2}}\vec{a}_{ij}$, $\pm\frac{1}{\sqrt{2}}\vec{b}_i$ and $\pm\frac{1}{\sqrt{2}}\vec{a}_i$ are the weight vectors of the 56-dimensional fundamental representation of E_7. To see this, we follow the same strategy as used in $D = 5$. First, we note that the dilaton vectors are given by (1.2) and (1.3), where the vectors $\vec{f}_i$ and $\vec{g}$ can be expressed as

$$\vec{f}_i = \sqrt{2}\vec{e}_i + \tfrac{1}{7}(3 - \sqrt{2})\sum_j \vec{e}_j \ , \qquad \vec{g} = \sum_i \vec{e}_i \ , \tag{3.9}$$

where $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. In terms of e_i, the simple roots of E_7 can be written as

$$\vec{\alpha}_1 = \vec{e}_2 - \vec{e}_3 \ , \qquad \vec{\alpha}_2 = \vec{e}_3 - \vec{e}_4 \ , \qquad \vec{\alpha}_3 = \vec{e}_4 - \vec{e}_5 \ , \qquad \vec{\alpha}_4 = \vec{e}_5 - \vec{e}_6 \ ,$$

$$\vec{\alpha}_5 = \vec{e}_5 + \vec{e}_6 \ , \qquad \vec{\alpha}_6 = \tfrac{1}{2}(\vec{e}_1 - \vec{e}_2 - \vec{e}_3 - \vec{e}_4 - \vec{e}_5 + \vec{e}_6) - \tfrac{1}{\sqrt{2}}\vec{e}_7 \ , \qquad \vec{\alpha}_7 = \sqrt{2}\vec{e}_7 \ . \tag{3.10}$$

The action of the Weyl group of E_7 can be generated by the seven Weyl reflections S_i in the hyperplanes orthogonal to the simple roots. The highest weight vector $\vec{\mu}$ of the 56-dimensional fundamental representation is defined by $\vec{\alpha}_i \cdot \vec{\mu} = \delta_{i7}$. The rest of the 55 weight vectors are obtained by acting with the Weyl group on the highest-weight vector. We find that the 28 dilaton vectors of the 2-form field strengths, together with their negatives, are $\sqrt{2}$ times the weight vectors of the 56-dimensional representation of E_7, after an orthogonal transformation of the basis for the unit vectors $\vec{e}_i$. For the 1-form field strengths, analogously to the higher-dimensional cases, the 63 dilaton vectors, together with their negatives, form a 126-component multiplet of the Weyl group of E_7; they correspond to the non-zero weights of the adjoint representation of E_7.

Finally, we consider maximal supergravity in $D = 3$, where there are a total of 120 1-form field strengths, corresponding to $\vec{a}_{ijk}$, $\vec{b}_{ij}$, $\vec{a}_{ij}^* = -\vec{a}_{ij}$ and $\vec{b}_i^* = -\vec{b}_i$. These vectors are given by (1.2) and (1.3), with $\vec{f}_i$ and $\vec{g}$ given by

$$\vec{f}_i = \sqrt{2}\vec{e}_i + \tfrac{1}{2\sqrt{2}}\sum_i \vec{e}_j \ , \qquad \vec{g} = \sqrt{2}\sum_i \vec{e}_i \ , \tag{3.11}$$

where $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. To see how the dilaton vectors transform under the action of the Weyl group of E_8, we note that the simple roots of E_8 can be written in terms of $\vec{e}_i$ as

$$\vec{\alpha}_1 = \vec{e}_2 - \vec{e}_3 , \qquad \vec{\alpha}_2 = \vec{e}_3 - \vec{e}_4 , \qquad \vec{\alpha}_3 = \vec{e}_4 - \vec{e}_5 ,$$

$$\vec{\alpha}_4 = \vec{e}_5 - \vec{e}_6 , \qquad \vec{\alpha}_5 = \vec{e}_6 - \vec{e}_7 , \qquad \vec{\alpha}_6 = \vec{e}_7 - \vec{e}_8 , \qquad (3.12)$$

$$\vec{\alpha}_7 = \vec{e}_7 + \vec{e}_8 , \qquad \vec{\alpha}_8 = \tfrac{1}{2}(\vec{e}_1 + \vec{e}_8 - (\vec{e}_2 + \cdots + \vec{e}_7)) .$$

It is now very easy to verify that the 120 dilaton vectors, together with their negatives, form a 240-component multiplet of the Weyl group of E_8. These vectors correspond to the non-zero weights of the adjoint representation of E_8.

Thus we see that the dilaton vectors in maximal supergravity theories in $3 \leq D \leq 5$ also form finite-order representations of the Weyl group of the symmetry group for the corresponding supergravity theory. In leading order, these U Weyl group symmetries correspond to interchanging the potentials (with certain sign changes) in parallel with their dilaton vectors. These transformations are discrete subgroups of the full symmetry groups of the supergravity theories. By the same argument as presented in the higher-dimensional cases, we can see that these Weyl groups are in fact symmetries of the corresponding supergravity theories.

4 S, T and X Weyl duality subgroups

In the previous two sections, we established that there is a discrete symmetry in D-dimensional supergravity, namely the Weyl group of the $E_{n(+n)}(\mathbb{R})$ symmetry group of the supergravity theory, where $n = 11 - D$. $E_{n(+n)}(\mathbb{Z})$ is the conjectured U duality of the associated string theory. This U duality contains various subgroups, which can be identified as S, T and X dualities. S duality is a conjectured non-perturbative symmetry of a string theory that that relates the strong and weak coupling regimes. In the theories we are considering, namely type II strings compactified on a torus, S duality exists only in $D = 4$, and has the effect of interchanging the twelve NS-NS 2-forms with their duals. It is an $SL(2, \mathbb{Z})$ subgroup of the U duality group $E_7(\mathbb{Z})$. T duality is a perturbative symmetry of string theory, valid order by order in the string coupling g, although non-perturbative in α'. It is an $SO(n-1, n-1; \mathbb{Z})$ subgroup of $E_{n(+n)}(\mathbb{Z})$ that acts on the internal dimensions of the compactified string, preserving the NS-NS and R-R sectors. X duality is another non-perturbative $SL(2, \mathbb{Z})$ subgroup of $E_{n(+n)}$, which interchanges the NS-NS and R-R fields. Note that unlike S duality, X duality exists in all dimensions $D \leq 10$ for type IIB strings and $D \leq 9$ for type IIA strings.

In this section, we shall study how the Weyl group of the U duality group $E_{n(+n)}$ decomposes into Weyl groups of the S, T and X duality groups. In order to do so, it is useful first to identify

which of the field strengths lie in the NS-NS sector, and which lie in the R-R sector of the type IIA string. In $D = 10$, the 3-form field strength is an NS-NS field, and the 4-form and 2-form field strengths are R-R fields. In lower dimensions, all fields that are derived from the 3-form or the metric are NS-NS fields, and all those derived from the 4-form or 2-form are R-R fields. Thus, in our notation, we have

$$
\begin{aligned}
\text{NS} - \text{NS}: \quad & \left\{ \begin{array}{cccccc} F_3^{(1)} & F_2^{(1\alpha)} & F_1^{(1\alpha\beta)} & \mathcal{F}_2^{(\alpha)} & \mathcal{F}_1^{(\alpha\beta)} \\ \vec{a}_1 & \vec{a}_{1\alpha} & \vec{a}_{1\alpha\beta} & \vec{b}_\alpha & b_{\alpha\beta} \end{array} \right. \\
\text{R} - \text{R}: \quad & \left\{ \begin{array}{cccccc} F_4 & F_3^{(\alpha)} & F_2^{(\alpha\beta)} & F_1^{(\alpha\beta\gamma)} & \mathcal{F}_2^{(1)} & \mathcal{F}_1^{(1\alpha)} \\ \vec{a} & \vec{a}_\alpha & \vec{a}_{\alpha\beta} & \vec{a}_{\alpha\beta\gamma} & \vec{b}_1 & \vec{b}_{1\alpha} \end{array} \right.
\end{aligned}
\tag{4.1}
$$

where we have decomposed the internal index i as $i = (1, \alpha)$, and we also indicate the dilaton vector associated with each field strength.

Let us begin by discussing T duality. The full T duality group is the $SO(n-1, n-1)$ subgroup of the $E_{n(+n)}$ U duality. The T Weyl group is the Weyl group of $SO(n-1, n-1)$, which is a subgroup of the U Weyl group that we have discussed in the previous sections. Since the dilaton vectors of the various field strengths form multiplets under the U Weyl group, we may examine how they decompose under the T Weyl subgroup. We shall show that dilaton vectors for the NS-NS or the R-R fields form independent multiplets under the T Weyl group. The details of the multiplet structures depend on the dimension of the theory. However, for the dilaton vectors $\vec{a}_{1\alpha}$ and $\vec{b}_\alpha$ for the $2(10 - D)$ NS-NS 2-forms, it can be discussed in general. They form a $2(10 - D)$-dimensional representation of $SO(10 - D, 10 - D)$ for $D \leq 8$. To see this, we note that it follows from (1.2) that these dilaton vectors can be expressed as

$$
\vec{a}_{1\alpha} = \sqrt{2}\,\vec{e}_\alpha + \tfrac{1}{2}(\vec{f}_1 - \vec{g}) \,, \qquad \vec{b}_\alpha = -\sqrt{2}\,\vec{e}_\alpha + \tfrac{1}{2}(\vec{f}_1 - \vec{g}) \,,
\tag{4.2}
$$

where $\vec{e}_\alpha = \frac{1}{2\sqrt{2}}(2\vec{f}_\alpha + \vec{f}_1 - \vec{g})$, which satisfy $\vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta}$. The vectors $\pm\vec{e}_\alpha$ are the weight vectors of the $2(10 - D)$-dimensional fundamental representation of $SO(10 - D, 10 - D)$. The action of the T Weyl group on the vectors $\vec{e}_\alpha$ is to permute any pair, with or without changing their signs. Since the vector $(\vec{f}_1 - \vec{g})$ is invariant under this action, it follows that the vectors $\vec{a}_{1\alpha}$ and $\vec{b}_\alpha$ form a $2(10 - D)$-component multiplet under the T Weyl group. This discussion breaks down in the case of $D = 9$, where there is no non-trivial T duality that interchanges field strengths. Having obtained the weight vectors of the fundamental representation of the T duality group, on which the Weyl group action can be simply stated, it is straightforward to study the multiplet structure of the remaining field strengths, since their dilaton vectors can be expressed in terms of linear combinations of $\vec{e}_\alpha$ and the T-invariant vector $(\vec{f}_1 - \vec{g})$. However, this is not the most convenient way to study the T Weyl group multiplets for all the field strengths. Since we have already obtained

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the U Weyl group multiplet structure in the previous sections, it is simpler just to read off the T duality structure from the U duality. For example, in $D = 7$ the U Weyl group S_5 permutes the five 3-form dilaton vectors $\vec{a}_I$, where $\vec{a}_5$ is the dilaton vector for the dualised 4-form field strength. The T Weyl group is the S_4 subgroup of S_5 generated by permutations of the 4 dilaton vectors $\vec{a}_\alpha$ with $\alpha = 2, 3, 4, 5$, *i.e.* it is the Weyl group of the T duality group D_3. It follows from (3.5) that the 6 dilaton vectors for the NS-NS 2-forms and 4 dilaton vectors for the R-R 2-forms comprise 6 and 4 component multiplets under S_4 respectively. The 6 dilaton vectors for NS-NS 1-forms, together with their negatives, form a 12-component multiplet under S_4, and the 4 dilaton vectors for R-R 1-forms comprise a 4-component multiplet. Their negatives form another 4-component multiplet. As discussed in the previous sections, for $D = 9, 8, 7$ and 6 the U Weyl group is generated by its action on the dilaton vectors $\vec{a}$ and $\vec{a}_i$ for the 4-form and 3-form field strengths, since these vectors correspond to the weight vectors of the fundamental representation of the U duality group. The T Weyl groups, on the other hand, are generated by the same action, but with the vector $\vec{a}_1$ omitted. (Thus in $D = 9$, the T Weyl group is just the identity.) For $D = 5, 4, 3$, we showed that the U Weyl groups are generated by the simple Weyl actions S_i, with the roots given by (3.8), (3.10) and (3.12). The T Weyl groups are simply generated by the same simple Weyl actions that permute the indices i with $i \neq 1$, *i.e.* they are generated by the simple Weyl reflections S_i with $i = 6$, 1 and 8 omitted in $D = 5, 4$ and 3 respectively.

In $D = 4$, in addition to the T duality that preserves the NS-NS and R-R sectors, there is an $SL(2, \mathbb{Z})$ S duality that also preserves the two sectors. The Weyl group of this S duality interchanges the set of 12 dilaton vectors for the NS-NS 2-form field strengths with their negatives. We summarise the T and S Weyl groups and their multiplet structures for field strengths in the following table:

Dim.	3-Forms		2-Forms		1-Forms		T	S	U
	NS-NS	R-R	NS-NS	R-R	NS-NS	R-R			
9	1	1	$1 + 1$	1	$-$	$1 + 1$	D_1	$-$	A_1
8	(1,1)	(2,1)	(2,2)	$(1,1) + (1,1)$	$(1,2) + (2,1)$	$(2,1) + (2,1)$	D_2	$-$	$A_2 \times A_1$
7	1	4	6	4	12	$4 + 4$	D_3	$-$	A_4
6	$1 + 1$	8	8	8	24	$8 + 8$	D_4	$-$	D_5
5			$10 + 1$	16	40	$16 + 16$	D_5	$-$	E_6
4			(12,2)	(32,1)	$(1,2) + (60,1)$	(32,2)	D_6	A_1	E_7
3					$14 + 14 + 84$	$64 + 64$	D_7	$-$	E_8

Table 1: T and S Weyl duality multiplets for type IIA strings

A number of comments on the table are in order. Firstly, we note that the various duality groups are given in terms of their Dynkin classification. Thus the T duality group in dimension D is $D_{n-1} \simeq SO(n-1, n-1)$, where $n = 11 - D$. The A_m groups appearing in the S and U duality columns denote the non-compact forms $SL(m+1)$. As symmetries of supergravity theories, all the groups considered are defined over the reals; as conjectured full string symmetries, they are defined over the integers. In $D = 8$, the T duality group is D_2, which is isomorphic to $A_1 \times A_1 \simeq SL(2) \times SL(2)$. The multiplets under the T Weyl group are specified by their dimensions under the two S_2 Weyl groups of these two $SL(2)$'s. The first is the subgroup of the S_3 factor in the full U Weyl group $S_3 \times S_2$ that corresponds to permutations just involving the $\vec{a}_i$ dilaton vectors with $i = 2$ and 3. The second S_2 is just the S_2 factor in $S_3 \times S_2$, corresponding to the interchange of the dilaton vector $\vec{a}$ and its negative. In $D = 4$, there is in addition an S duality symmetry. This arises as the A_1 factor in the decomposition of the U duality group down to the T duality group; $E_7 \longrightarrow D_6 \times A_1$. The orders of the multiplets in $D = 4$ are of the form (p, q), where p is the order under the T Weyl group of D_6, and q is the order under the S Weyl group of A_1. Note that the occurrence of an extra A_1 factor in the decomposition of the U duality group to the T duality group is unique to $D = 4$, and so toroidal compactifications of the type IIA string exhibit S duality only for $D = 4$.

Finally, we remark that there is an X duality in every dimension, which has the effect of mapping fields between the NS-NS and R-R sectors. It is an $SL(2)$ symmetry whose S_2 Weyl group is generated by the Weyl reflection corresponding to the simple root of the U duality group that is truncated in the passage to the T duality subgroup. Combined with the $SL(n-1)$ subgroup of the T duality group that permutes the indices $\alpha = \{2, 3, \ldots, n\}$ on the dilaton vectors, the X duality has the effect of interchanging the index $i = 1$ with any index $i = \alpha$, thus corresponding to an interchange of NS-NS and R-R sectors. It is worth remarking T and X dualities do not commute, and that their closure generates the entire U duality. Note that we have not included the dilaton vector $\vec{a}$ for the 4-form field strength in the table, since it would appear explicitly only in $D \geq 8$. In fact in $D = 9$, it is a singlet under U duality. In $D = 8$, it, together with its negative, is a $(1, 2)$ under both T and U Weyl dualities. Under X duality, the dilaton vector $\vec{a}$ and its negative are both singlets.

5 p-brane U Weyl group multiplets

In the previous two sections, we established that there is a discrete symmetry W in D-dimensional supergravity that may be identified with the Weyl group of the $G \simeq E_{n(+n)}$ symmetry group

for the supergravity theory, where $n = 11 - D$. Now, we shall study how p-brane solutions form multiplets under the action of the Weyl group. First we shall give a short review of p-brane solutions in maximal supergravity theories, and then we shall discuss their supersymmetry properties. In particular, we shall show that the supersymmetry properties of a p-brane solution are invariant under the action of the U Weyl group. After this, we shall discuss the p-brane U Weyl group multiplets.

5.1 p-brane solutions

In D-dimensional supergravity, one can construct a p-brane solution with N participating n-form field strengths, with $N \leq (11 - D)$. The relevant part of the Lagrangian is given by

$$\mathcal{L} = eR - \tfrac{1}{2}e\,(\partial\vec{\phi})^2 - \frac{1}{2n!}e \sum_{b=1}^{N} e^{\vec{c}_b \cdot \vec{\phi}} F_b^2 \ , \tag{5.1}$$

where $\vec{c}_b$ denotes the dilaton vectors associated with the N participating field strengths F_a. In the single-scalar p-brane solutions, the Lagrangian (5.1) is consistently truncated to the simple form

$$\mathcal{L} = eR - \tfrac{1}{2}e\,(\partial\phi)^2 - \frac{1}{2n!}e\,e^{a\phi} F^2 \ , \tag{5.2}$$

where the scalar field ϕ is some linear combination of the dilatonic scalars $\vec{\phi}$ of the D-dimensional theory, and F is a single canonically-normalised n-index field strength, to which all of the N original field strengths that participate in the solution are proportional. The constant a appearing in the exponential prefactor can be conveniently parameterised as

$$a^2 = \Delta - \frac{2d\tilde{d}}{D-2} \ , \tag{5.3}$$

where $\tilde{d} \equiv D - d - 2$ and $d\tilde{d} = (n-1)(D-n-1)$. The quantity Δ, unlike a itself, is preserved under Kaluza-Klein dimensional reduction [11]. The value of a^2 and the ratios of the squares of the field strengths F_b^2 are determined by the dot products of the dilaton vectors $\vec{c}_b$. If the matrix $M_{ab} = \vec{c}_a \cdot \vec{c}_b$ is invertible, they are given by [9]

$$F_b^2 = a^2 \sum_c (M^{-1})_{bc} F^2 \ , \qquad a^2 = \Big(\sum_{b,c} (M^{-1})_{bc} \Big)^{-1} \ . \tag{5.4}$$

If M_{bc} is non-invertible, there is always a solution with $a = 0$, where the squares of the field strengths F_b^2 are equal to each other. It turns out that in this singular case, this is the only solution that does not simply reduce to an already-considered non-singular case with a smaller number N of participating field strengths [9]. Clearly, if the number of participating field strengths exceeds the dimension $(11 - D)$ of the dilaton vectors, then the associated matrix M_{bc} will be singular, and in

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fact it turns out that in all such cases, there is no new solution [9]. Thus in any dimension D, it follows that the number N of participating field strengths must always satisfy $N \leq 11 - D$.

Having reduced the Lagrangian (1.1) to (5.2) by the above procedure, it is now a simple matter to obtain solutions for the equations of motion that follow from (5.2). The metric ansatz for p-brane solutions is

$$ds^2 = e^{2A} \, dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} \, dy^m dy^m \, , \tag{5.5}$$

where A and B are functions only of $r = \sqrt{y^m y^m}$. The coordinates x^μ lie in the d-dimensional world volume of the p-brane, and y^m lie in the $(D-d)$-dimensional transverse space. The ansatz for the field strength F is either

$$F_{m\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}}(e^C)' \frac{y^m}{r} \qquad \text{or} \qquad F_{m_1\cdots m_n} = \lambda \epsilon_{m_1\cdots m_n p} \frac{y^p}{r^{n+1}} \, . \tag{5.6}$$

The first choice yields an elementary $(n-2)$-brane carrying an electric charge $u = \frac{1}{4\omega_{D-n}} \int_{\partial\Sigma}^* F$, whilst the second choice yields a solitonic $(D-n-2)$-brane carrying a magnetic charge $v = \frac{1}{4\omega_n} \int_{\partial\Sigma} F$, where $\partial\Sigma$ is the boundary $(D-d-1)$-sphere of the transverse space with volume ω_{D-d-1}. The solutions for both the elementary and solitonic p-branes take the form

$$\begin{aligned}
ds^2 &= \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-\frac{4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{\frac{4d}{\Delta(D-2)}} dy^m dy^m \, , \\
e^\phi &= \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{\frac{2a}{\epsilon\Delta}} \, ,
\end{aligned} \tag{5.7}$$

where $\epsilon = 1$ and -1 for the elementary and the solitonic solutions respectively, and $k = -\sqrt{\Delta}\lambda/(2\tilde{d})$. In the elementary case, the function C satisfies the equation

$$e^C = \frac{2}{\sqrt{\Delta}}\left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-1} \, . \tag{5.8}$$

The masses of these solutions are given by $\lambda/(2\sqrt{\Delta})$. Note that the dual of the field strength in the elementary case is identical to the field strength of the solitonic case, and *vice versa*. For this reason, we shall only consider solutions for field strengths with $n \leq D/2$.

One way to enumerate all the single-scalar p-brane solutions of this type is to consider, for each dimension D and each degree n for the field strengths, all possible choices of the associated $N \leq 11 - D$ dilaton vectors, and then calculate the values of a, and the corresponding ratios of participating field strengths, using the above equations. Although this is easily done for $n = 4$ (where there is always only one field strength) and for $n = 3$ (where the number of field strengths is small), for $n = 2$ and $n = 1$ the numbers of field strengths grow significantly with decreasing dimension D and so the enumeration is most conveniently carried out by computer. The p-brane solutions are characterised by the values of Δ and the ratios of the participating field strengths,

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which are determined by the set of dot products of the associated dilaton vectors. Under the action of the Weyl group, which preserves these dot products, the characteristics of the solutions are therefore preserved. Thus by acting with the Weyl group on a given p-brane solution, an entire multiplet of solutions with the same value Δ and ratios of field strengths is generated.

5.2 Supersymmetry and the U Weyl group

Another important characteristic of a p-brane solution is the fraction of 11-dimensional supersymmetry that is preserved by it. We shall show explicitly that the U Weyl group also leaves this characteristic invariant. In order to determine the supersymmetry properties of the various p-brane solutions, it suffices to study the transformation laws of $D = 11$ supergravity. In particular, from the commutator of the conserved supercharges $Q_\epsilon = \int_{\partial\Sigma} \bar{\epsilon}\Gamma^{ABC}\psi_C d\Sigma_{AB}$, we may read off the 32×32 Bogomol'nyi matrix $\mathcal{M}$, defined by $[Q_{\epsilon_1}, Q_{\epsilon_2}] = \epsilon_1^\dagger \mathcal{M}\epsilon_2$, whose zero eigenvalues correspond to unbroken generators of $D = 11$ supersymmetry. The expression for $\mathcal{M}$ for maximal supergravity in an arbitrary dimension D then follows by dimensional reduction of the expression in $D = 11$. A straightforward calculation shows that it is given by [9]

$$\begin{aligned}
\mathcal{M} &= m\mathbb{1} + u\,\Gamma_{012} + u_i\,\Gamma_{01i} + \tfrac{1}{2}u_{ij}\,\Gamma_{0ij} + \tfrac{1}{6}u_{ijk}\Gamma_{ijk} + p_i\Gamma_{0i} + \tfrac{1}{2}p_{ij}\Gamma_{ij} \\
&\quad + v\,\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}} + v_i\,\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}i} + \tfrac{1}{2}v_{ij}\,\Gamma_{\hat{1}\hat{2}\hat{3}ij} + \tfrac{1}{6}v_{ijk}\,\Gamma_{\hat{1}\hat{2}ijk} + q_i\,\Gamma_{\hat{1}\hat{2}\hat{3}i} + \tfrac{1}{2}q_{ij}\,\Gamma_{\hat{1}\hat{2}ij}\,.
\end{aligned} \tag{5.9}$$

The indices $0, 1, \ldots$ run over the dimension of the p-brane worldvolume, $\hat{1}, \hat{2}, \ldots$ run over the transverse space of the y^m coordinates, and $i, j, \ldots$ run over the dimensions compactified in the Kaluza-Klein reduction from 11 to D dimensions. The quantities u, u_i, u_{ij}, u_{ijk}, p_i and p_{ij} are the electric charges, and v, v_i, v_{ij}, v_{ijk}, q_i and q_{ij} are the magnetic charges, associated with F_4, F_3^i, F_2^{ij}, F_1^{ijk}, $\mathcal{F}_2^i$ and $\mathcal{F}_1^{ij}$ respectively. These magnetic and electric charges are given by integrals of the corresponding field strengths and their duals respectively, together with Chern-Simons corrections, over the boundary of the transverse space of the solution [13, 14]. The quantity m is the mass per unit p-volume for the solution, given by $\frac{1}{2}(A' - B')e^{-B}r^{\tilde{d}+1}$ in the limit $r \to \infty$, where $\tilde{d} \equiv D - d - 2$.

Once the mass per unit p-volume and the Page charges have been determined for a given p-brane solution, it becomes a straightforward algebraic exercise to substitute the results for the Page charges into the Bogomol'nyi matrix (5.9) and to calculate the eigenvalues. The fraction of $D = 11$ supersymmetry that is preserved by a solution is then equal to $k/32$, where k is the number of zero eigenvalues of the 32×32 Bogomol'nyi matrix. We shall show that the Bogomol'nyi matrix is invariant under the action of the U Weyl group, and that p-brane solutions with the same set of eigenvalues form representations of the U Weyl group.

In our discussion of the U Weyl groups in all dimensions $D \le 9$, we saw that they always have a

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subgroup S_{11-D} consisting simply of the permutations of the internal compactified dimensions, *i.e.* $i \leftrightarrow j$ for any i and j. This symmetry can be easily seen in the Bogomol'nyi matrix by considering the transformation $\mathcal{M} \longrightarrow (L_{ij})^{-1}\mathcal{M}L_{ij}$, where

$$L_{ij} = e^{\frac{\pi}{4}\Gamma_{ij}} = \tfrac{1}{\sqrt{2}}(\mathbb{1} + \Gamma_{ij}) \ . \tag{5.10}$$

This transformation interchanges the internal index values i and j on the Γ matrices appearing in $\mathcal{M}$ (with sign changes in some cases). Thus the Bogomol'nyi matrix is invariant provided that the indices of the Page charges undergo appropriate conjugate transformations. In the case of $D = 9$, where the non-trivial symmetry group is $SL(2,\mathbb{R})$, this S_2 permutation symmetry is in fact the same as the entire Weyl-group symmetry of the theory. Thus the Bogomol'nyi matrix, and hence the supersymmetry properties of p-brane solutions, are invariant under the action of the U Weyl group. It is easy to check that the sign changes of the Page charges are precisely consistent with the sign changes of the potentials given in (2.10).

In lower dimensions, $D \leq 8$, the U Weyl group is larger than the S_{11-D} permutation group, due to additional discrete symmetries that involve the interchange of certain field strengths and the Hodge duals of other field strengths. The rank of the supergravity symmetry group is $(11-D)$, which implies that its Weyl group should be generated by $(11-D)$ group elements. We have already obtained $(10-D)$ independent generators, namely $L_{i,i+1}$ with $1 \leq i \leq (10-D)$, from which the entire set of L_{ij} group elements can be constructed. The remaining independent generator can be taken to be L_{123}, where L_{ijk} is defined as[1]

$$L_{ijk} = e^{\frac{\pi}{4}\Gamma_{ijk}} = \tfrac{1}{\sqrt{2}}(\mathbb{1} + \Gamma_{ijk}) \ . \tag{5.11}$$

It is straightforward to establish that the Bogomol'nyi matrix is also invariant under the transformation $\mathcal{M} \longrightarrow (L_{ijk})^{-1}\mathcal{M}L_{ijk}$, provided again that the necessary conjugate transformations of the Page charges are performed. The complete finite group of transformations generated by $L_{i,i+1}$ and L_{123} is in fact isomorphic to the U Weyl group discussed previously, now implemented on the Γ matrices appearing in the Bogomol'nyi matrix. This can be illustrated simply by concentrating, in each dimension $D \leq 8$, on the group action on the set of Page charges of the highest-degree field strengths in that dimension.

In $D = 8$, the transformations generated by L_{ij} form the S_3 subgroup of the $S_3 \times S_2$ Weyl group. On the other hand, L_{123} interchanges Γ_{012} and $\Gamma_{\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}}$ in the Bogomol'nyi matrix, and therefore

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[1]In each dimension D, the matrices Γ_{ij} and Γ_{ijk} form some (or in $D \geq 6$ all) of the generators of the automorphism group of the associated Clifford algebra. Thus in $D = 9$, Γ_{12} alone closes on $SO(2)$, whilst in $D = 8, 7$ and 6, Γ_{ij} and Γ_{ijk} close on $SO(3) \times SO(2)$, $SO(5)$ and $SO(5) \times SO(5)$ respectively. In $D = 5$ and $D = 4$, we must include $\Gamma_{ijk\ell mn}$ as well, thereby achieving closure on $USp(8)$ and $SU(8)$ respectively. In $D = 3$ we must add the further generators $\Gamma_{ijk\ell mnp}$ too, thereby achieving closure on $SO(16)$.

corresponds to the S_2 symmetry that interchanges the electric and magnetic charges of the 4-form field strength. It is easy to verify that the lower-degree Page charges also transform in the proper way, as dictated by the Weyl group. In $D = 7$, L_{ijk} can also be expressed as $L_{i8} = e^{\frac{\pi}{4}\Gamma_{i8}}$ where Γ_8 is the product $\prod_\mu \Gamma_\mu \prod_m \Gamma_m$ of the $D = 11$ Γ matrices with 7-dimensional spacetime indices. The action of L_{i8} on the Γ matrices corresponds to interchanging the Page charge of the dual of the 4-form field strength F_4 with the Page charge of the 3-form field strength F_3^i. Thus the 4-form and 3-form sector of the Bogomol'nyi matrix is invariant under the extended S_5 Weyl group of $SL(5, \mathbb{R})$. A more detailed analysis shows that the lower-degree sectors of the Bogomol'nyi matrix are also invariant under the S_5 Weyl group. In $D = 6$, the group elements L_{ijk} can be expressed as L_{ij7}, where Γ_7 is the product of the $D = 11$ Γ matrices with 6-dimensional spacetime indices. The action of L_{ij7} is to interchange the electric and magnetic charges of the field strengths F_3^i and F_3^j. Together with the action of L_{ij}, this has the effect of interchanging the dilaton vectors $\vec{a}_i$ and $\vec{a}_j$ with or without sign change. This is precisely the action of the Weyl group of $D_5 \simeq SO(5,5)$, as discussed earlier.

The analysis of Weyl group invariance of the Bogomol'nyi matrix is more complicated in the lower dimensions $D \leq 5$, since now the symmetry groups are enlarged to the exceptional groups E_6, E_7 and E_8. In $D = 5$ and $D = 4$, we can choose one of the Γ matrices appearing in $\mathcal{M}$ that corresponds to any one of the Page charges for 2-forms, and fill out the entire set of 27 or 56 such Γ matrices respectively, by acting with the generators $L_{i,i+1}$ and L_{123}. Since this translates into a conjugate action on the Page charges, it follows that if we start from one 2-form Page charge, we can fill out the complete 27 or 56-component multiplets. This is consistent with what we saw in the previous section, where we showed that the dilaton vectors for the 2-form field strengths in $D = 5$ and $D = 4$ form the 27 and 56 irreducible representations of the relevant Weyl groups. In $D = 3$, only 1-form field strengths need be considered, since 2-forms are dual to 1-forms. In total there are 120 1-forms. So far, we have avoided discussing explicitly the Weyl group action on the 1-form sector of the Bogomol'nyi matrices, since the 0-form potentials undergo a more complicated non-linear transformation. In the previous section we saw that the full set of ℓ dilaton vectors for the 1-forms, together with their negatives, form a 2ℓ-component multiplet of the Weyl group. The reason why these dilaton vectors can change sign is quite different from the sign changes that we have seen for the dilaton vectors of the higher-degree forms in the special dimensions where their duals are of the same degree as the forms themselves. In those cases, changing the sign of a dilaton vector was associated with an interchange of the electric and magnetic charges of the associated field strength. Here, on the other hand, the sign change of the dilaton vector $\vec{c}$ for a 1-form field strength can occur in any dimension, and is a consequence of the non-linear nature of

the transformation for the 0-form potential, $A_0 \longrightarrow A_0' \sim e^{\vec{c}\cdot\vec{\phi}}A_0 + \cdots$, which compensates for the sign change of the dilaton prefactor in the Lagrangian. The Page charge remains invariant under this transformation, and so the set of ℓ 1-form Page charges form an ℓ-component multiplet of the Weyl group, even though their dilaton vectors form a (2ℓ)-component multiplet. This is consistent with the fact that in all dimensions, the action of the generators $L_{i,i+1}$ and L_{123} on any one of the Γ matrices for the 1-forms fills out the full set of Γ matrices for the 1-form field strengths.

5.3 p-brane U Weyl group multiplets

The procedure for obtaining a p-brane multiplet of the U Weyl group is now very simple. Consider a p-brane solution involving N field strengths F_a whose dilaton vectors are $\vec{c}_a$ with $c = 1, \cdots, N$. Then acting with the U Weyl group on the set of vectors $\{\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_N\}$ generates a full representation of the Weyl group, where each set of dilaton vectors has the same dot products as the original set. It is then straightforward to identify the associated sets of participating field strengths. The p-brane solutions constructed with these sets of field strengths form a multiplet under the U Weyl group.

The simplest multiplet structures occur for p-brane solutions involving only one field strength. These solutions all have $\Delta = 4$, and preserve $\frac{1}{2}$ of the 11-dimensional supersymmetry. In $D = 9$, the p-brane solutions using F_4 or $F_2^{(12)}$ are singlets, while those using F_3^i or $\mathcal{F}_2^i$ are doublets. The p-brane solution using $\mathcal{F}_1^{12}$ and its Weyl group partner (where the sign of its dilaton vector is reversed, and the 0-form potential undergoes a non-linear transformation) also form a doublet. The analogous multiplet structures can also be obtained in lower dimensions. For example, in $D = 4$, acting with the Weyl group of E_7 on any purely electric or purely magnetic 0-brane solution fills out a 56-component multiplet. In fact these are the 56 purely electric or purely magnetic black hole solutions in $D = 4$, whose significance was addressed in [3]. Analogously, acting on any string solution using a 1-form field strength fills out a 126-component multiplet of the Weyl group of E_7. This representation corresponds to the Weyl group action on the 126 non-zero weights of the adjoint representation of E_7. (Note that solutions related simply by sign changes of the electric or magnetic charges are regarded as being equivalent.)

The U Weyl group multiplet structures of p-brane solutions involving more than one field strength are more complicated. For solutions with $N = 2$ field strengths, there are two possible values of Δ, namely $\Delta = 2$ and $\Delta = 3$. The solutions with $\Delta = 2$ preserve $\frac{1}{4}$ of the supersymmetry, whilst those with $\Delta = 3$ break all the supersymmetry. The solutions with different Δ values cannot lie in the same multiplet since the dot products of the participating dilaton vectors are different. This is consistent with our discussion in the previous section, where we showed that the Weyl group preserves the Bogomol'nyi matrix, and hence it preserves the supersymmetry properties of

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the solutions. In $D = 9$, there is just one p-brane solution involving the two 3-form field strengths. This solution has $\Delta = 3$. There is also a singlet $\Delta = 3$ solution using the two 2-forms $\mathcal{F}_2^i$, and a doublet of $\Delta = 2$ solutions using $F_2^{(12)}$ together with either of the $\mathcal{F}_2^i$. Another example is $D = 4$. We find in this case that there are a total of 1512 solutions, half of which have $\Delta = 2$, and the other half have $\Delta = 3$.

With increasing numbers of participating field strengths, the possible values of Δ proliferate. The discussion of the multiplet structures for these solutions becomes tedious, albeit straightforward. However, most of these solutions are non-supersymmetric, and we shall not discuss them here in detail. The supersymmetric solutions have been classified in [9]. They have $\Delta = \frac{4}{N}$ for solutions with N participating field strengths, with the corresponding Page charges having equal magnitudes. There is only one 4-form field strength, which exists for $D \geq 8$ (in the sense that in dimensions lower than 8, it will be dualised to a lower-degree form). The corresponding p-brane solutions are singlets for $D \geq 8$, and form a doublet in $D = 8$. The 3-form field strengths exist for $D \geq 6$. The supersymmetric solutions all involve one field strength only, and thus have $\Delta = 4$. They form irreducible multiplets of order $2, 3, 5$ and 10 under the relevant Weyl groups in $D = 9, 8, 7$ and 6 respectively. The p-brane solutions involving more than one field strength are all non-supersymmetric, and the values of Δ are given by $\Delta = 2 + \frac{2}{N}$, with N from 2 up to 5. The multiplet structures of the 3-form p-brane solutions are presented in the following table:

Dim.	$\Delta = 4$	$\Delta = 3$	$\Delta = \frac{8}{3}$	$\Delta = \frac{5}{2}$	$\Delta = \frac{12}{5}$
$D = 9$	2	1			
$D = 8$	3	3	1		
$D = 7$	5	10	10	1	
$D = 6$	10	40	80	80	32

Table 2: U Weyl multiplets for 3-form solutions

The 2-form field strengths exist for $D \geq 4$. The supersymmetric solutions can involve up to 4 participating field strengths. The multiplet structure for these is given in the table below:

Dim.	$\Delta = 4$	$\Delta = 2$	$\Delta = \frac{4}{3}$	$\Delta = 1$
$D = 9$	$1 + 2$	2		
$D = 8$	6	6		
$D = 7$	10	15		
$D = 6$	16	40		
$D = 5$	27	135	45	
$D = 4$	56	756	2520	630

Table 3: U Weyl multiplets for supersymmetric 2-form solutions

The 1-form field strengths exist for all $3 \leq D \leq 9$. The supersymmetric solutions can involve up to 8 participating field strengths. The multiplet structure for them is given in the table below:

Dim.	$\Delta = 4$	$\Delta = 2$	$\Delta = \frac{4}{3}$	$\Delta = 1'$	$\Delta = 1$	$\Delta = \frac{4}{5}$	$\Delta = \frac{2}{3}$	$\Delta = \frac{4}{7}$	$\Delta = \frac{1}{2}$
$D = 9$	2								
$D = 8$	8	12							
$D = 7$	20	60							
$D = 6$	40	280	480	240					
$D = 5$	72	1080	4320	2160					
$D = 4$	126	3780	30240 +2520	15120	60480	90720	60480	17280	
$D = 3$	240	15120	302400	*	*	*	*	*	*

Table 4: U Weyl multiplets for supersymmetric 1-form solutions

Note that in $D = 4$, the 1-form solutions with $\Delta = \frac{4}{3}$ form two distinct irreducible multiplets under the U Weyl group, even though they have the same amount of preserved supersymmetry. Such a phenomenon occurs also for the 2-form solutions with $\Delta = 4$ in $D = 9$. In all other cases, the U Weyl group multiplets for supersymmetric solutions occur in single irreducible multiplets. Note also that there are two different 1-form solutions with $\Delta = 1$ in $D = 4$, which we denote by $\Delta = 1'$ and $\Delta = 1$ [9]. The former preserves $\frac{1}{8}$ of the supersymmetry and the latter preserves $\frac{1}{16}$. They form different irreducible multiplets, as indicated in table 4.

The multiplicities for 1-form solutions in $D = 3$ are large, and we have obtained results only for $N \leq 3$ field strengths; the entries containing a $*$ in table 4 correspond to solutions whose multiplicities have not yet been determined. Since there are 120 1-form field strengths, it follows that $\Delta = 4$ solutions form a 240-component multiplet of the Weyl group of E_8. There can be up to

$N = 8$ participating field strengths, and $\Delta = \frac{4}{N}$, with the new value $\Delta = \frac{1}{2}$ occurring when $N = 8$. The Bogomol'nyi matrices, and hence the amount of preserved supersymmetry, for the solutions with $N \leq 7$ are the same as those in $D = 4$, as discussed in [9]. For the supersymmetric solutions with 8 field strengths, whose dilaton vectors satisfy $\vec{c}_a \cdot \vec{c}_b = 4\delta_{ab}$, we find that the solution preserves $\frac{1}{16}$ of the supersymmetry.

The above discussion of the Weyl-group multiplet structure for p-brane solutions can be applied not only to the single-scalar solutions with purely electric or purely magnetic charges, but also to multi-scalar solutions, and to dyonic solutions in even dimensions. Each supersymmetric solution involving N field strengths can be generalised to an N-scalar solution, in which the N Page charges become independent parameters [12]. These multi-scalar solutions form a bigger multiplet than the single-scalar ones, since the Page charges are now independent and hence the participating field strengths are distinguishable. For example, for solutions with N 2-form field strengths, the dimensions of the Weyl group multiplets are increased by a factor of $N!$. In $D = 3$, a new supersymmetric solution arises with 8 field strengths, which has not been discussed previously. The metric of this multi-scalar solution is given by the general results in [12]. We find that the eigenvalues of the Bogomol'nyi matrix are given by

$$
\begin{aligned}
\mu \;=\; & 2\{0, \lambda_{1245}, \lambda_{1346}, \lambda_{2356}, \lambda_{1237}, \lambda_{3457}, \lambda_{2467}, \lambda_{1567}, \lambda_{2348}, \lambda_{1358}, \\
& \lambda_{1268}, \lambda_{4568}, \lambda_{1478}, \lambda_{2578}, \lambda_{3678}, \lambda_{12345678}\} \,,
\end{aligned}
\tag{5.12}
$$

where $\lambda_{a\cdots b} = \lambda_a + \cdots + \lambda_b$, and $\frac{1}{4}\lambda_a$ are the eight independent Page charges for the field strengths, for example $\mathcal{F}_1^{(12)}$, $\mathcal{F}_1^{(34)}$, $\mathcal{F}_1^{(56)}$, $F_1^{(127)}$, $F_1^{(347)}$, $F_1^{(567)}$, $^*F_2^{(78)}$ and $^*\mathcal{F}_2^{(8)}$. Each eigenvalue in (5.12) has degeneracy 2. Note that the last of the eigenvalues (5.12) is twice the mass of the p-brane solution. When all the λ_a are the same, the solution reduces to the single-scalar solution with $\Delta = \frac{1}{2}$, and the eigenvalues become $\mu = m\{0_2, 1_{28}, 2_2\}$. Thus it preserves $\frac{1}{16}$ of the supersymmetry, as for the case with generic values of the Page charges. There are further supersymmetry enhancements for special values of the charges $\lambda_a/4$, as can be seen from (5.12). In some cases, the Bogomol'nyi matrix has indefinite signature while in other cases all the non-zero eigenvalues are positive. As in some of the previous p-brane solutions with $N \geq 4$ field strengths, the supersymmetry of this $N = 8$ solution is also sensitive to the choices of signs for the Page charges $\lambda_a/4$, with half of the sign choices giving the supersymmetric solution we described above. The other half of the choices give rise to solutions that are non-supersymmetric for generic values of the Page charges. The eigenvalues of the Bogomol'nyi matrix are given by

$$
\begin{aligned}
\mu \;=\; & 2\{\lambda_8, \lambda_{234}, \lambda_{135}, \lambda_{126}, \lambda_{456}, \lambda_{147}, \lambda_{257}, \lambda_{367}, \lambda_{12458}, \lambda_{13468}, \\
& \lambda_{23568}, \lambda_{12378}, \lambda_{34578}, \lambda_{24678}, \lambda_{15678}, \lambda_{1234567}\} \,.
\end{aligned}
\tag{5.13}
$$

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Note that for generic values of the charges, none of the eigenvalues is proportional to the mass.

Now let us consider the dyonic solutions, which exist in $D = 8, 6$ and 4. There are two kinds of dyonic solutions. In the first type, although there are both electric and magnetic charges, each participating field strength carries either electric charge or magnetic charge, but not both. These types of solutions are already present in the multiplets discussed above, in $D = 6$ and 4 for the cases with more than one field strength. This is simply because the dilaton vectors for 3-forms in $D = 6$ (and 2-forms in $D = 4$), together with their negatives (corresponding to dualisation of the fields), form an irreducible multiplet under the Weyl group. Thus if we start with a solution with purely electric or purely magnetic charges, the action of the Weyl group will map it to a set of solutions including some with both electric and magnetic charges. In the dyonic solutions of the second type, each participating field strength can be both electric and magnetic. In $D = 8$, such a dyonic membrane solution, which involves a non-vanishing axion also, was constructed in [15].[2] The action of the S_2 factor of the $S_3 \times S_2$ Weyl group is to interchange the electric and magnetic charges of the 4-form field strength, and hence they form a doublet. In $D = 6$, dyonic solutions involving only one 3-form field strength, and with no non-vanishing axions, form a 10-component multiplet under the Weyl group. Dyonic solutions involving more than one 3-form field strength have not yet been constructed. These solutions in general would be non-supersymmetric since the corresponding solutions with purely electric or purely magnetic charges are non-supersymmetric. In $D = 4$, examples of dyonic solutions of the second type with $N = 2$ and $N = 4$ field strengths have been constructed in [9]. These solutions involve no non-vanishing axions. The U Weyl group action on these solutions generates multiplets of orders 756 and 630, just as for the purely electric or purely magnetic solutions. Some members of the multiplets have non-vanishing axions. Dyonic solutions of the second type with $N = 1$ and 3 field strengths have not yet been constructed, since all those solutions seem to involve non-vanishing axions. The analysis of their multiplet structure would, however, be straightforward.

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[2]Commonly, when p-brane solutions are constructed in the literature, the equations of motion are simplified by considering cases where the $\mathcal{L}_{FFA}$ terms and the Chern-Simons modifications to the field strengths make no contribution. In the multiplets of solutions that we have discussed, it sometimes happens that some members of a multiplet involve field configurations for which these contributions do not vanish. In these cases, certain 0-form potentials become non-vanishing. Note, however, that the associated 1-form field strengths do not carry any electric or magnetic charge.

6 Conclusions

In this paper, we have studied the Weyl groups of the U duality groups for the type IIA string compactified to D dimensions on a torus. The U Weyl group describes a discrete symmetry of the theory that corresponds to certain permutations of the field strengths, and in some cases their duals. It is analogous to the discrete $\mathbb{Z}_2$ subgroup of the $U(1)$ duality symmetry in electromagnetism, which interchanges the roles of B and E, and which captures the essence of the electric-magnetic duality. Although the U duality group, like its discrete Weyl group, preserves the isotropicity of a p-brane solution, the full U duality group will transform a solution with N participating field strengths into a solution with more than N field strengths involved, and with a more complicated axion configuration. This complexity simply reflects the fact that an intermediate rotation has been performed, in which the resulting configuration is no longer aligned cleanly along the axes in the space of field strengths.

We have also discussed the Weyl groups of various subgroups of the U duality group, namely the S, T and X duality subgroups. The Weyl groups of the S, T and X duality groups single out the subsets of these transformations that keep the solutions aligned along the axes in the space of field strength tensors.

We have seen that the U Weyl groups may also be identified as the stability duality groups of the vacuum. This is most easily seen by restricting attention to the standard vacuum, where the scalar fields asymptotically tend to zero. For this standard vacuum, the full U duality group $G(\mathbb{Z}) \simeq E_{n(+n)}(\mathbb{Z})$ is simply represented by integer-valued matrices. This identification is made without implying a commitment to any particular interpretation of duality symmetries as either ordinary global symmetries, or as effectively local discrete symmetries, to be divided out in constructing the string spectrum. If one chooses the former view, dualities may be seen as generators of multiplets of distinct, *i.e.* non-identified, p-brane solutions in the fashion of Ref. [3]. In this view, the action of the $G(\mathbb{Z})$ transformations on p-brane solutions generally needs to be followed by an analytic continuation of the moduli (*i.e.* the asymptotic values of the scalar fields) back to their standard values, so as to remain within the original vacuum sector. This analytic continuation of the moduli does not preserve the tensions (*i.e.* the masses/unit p-volume) of the solutions, giving rise to a $G(\mathbb{Z})$ multiplet of solutions at different tensions. However, since the Weyl subgroup of G preserves the standard vacuum, no such analytic continuation is necessary. Thus, the Weyl group may also be seen as the symmetry group of p-brane solutions at fixed tension, within a given vacuum sector.

We have seen that the U Weyl group preserves the number of field strengths that participate in a given solution. A remark should be made about the role of the axions at this point. Axions play two different roles in the various p-brane solutions. Firstly, an axion can arise as a 0-form potential

for a 1-form field strength carrying an electric or magnetic charge, giving rise to an instanton or $(D-3)$-brane solution. Secondly, an axion can instead arise as a field that is required to be non-zero in a solution where some other field strengths carry electric and/or magnetic charges. In this latter case, the field strength of the axion itself does not carry any electric or magnetic charge. In a solution with only one higher-degree ($n \geq 2$) field strength, all the axions vanish, and they remain zero under any U Weyl group transformation. If the solution involves only one 1-form field strength, the U Weyl group transforms it to a solution with another 1-form field strength, with the rest of the axions vanishing. The axions play their second role in solutions with more than one field strength, after acting with the U Weyl group transformation; however, as we stated above, the corresponding 1-form field strengths do not carry electric or magnetic charges. Thus to be precise, the U Weyl group preserves the number of non-vanishing charges in a p-brane solution, while the full U duality group does not.

The results of this paper may be combined with the various processes of dimensional reduction and oxidation to establish relations between 'brane solutions of different dimensionality p. There are two basic types of dimensional reduction relevant to the classification of p-branes. One is diagonal dimensional reduction [16, 11], proceeding by identification of points along the orbits of isometries of the solutions tangential to the worldvolume. The second is "straight" dimensional reduction [3, 17], employing the fact that p-brane solutions may be "stacked up" owing to their zero-force properties. Both procedures preserve the Δ values (5.3) of p-brane solutions [11, 17]. Then combining these different reduction/oxidation procedures with duality transformations permits relations to be established between different p-branes. For example, combining a straight reduction from $D + 10$ to $D = 6$, a duality transformation in $D = 6$ and a diagonal oxidation from $D = 6$ back to $D = 10$ relates the $D = 10$ string and 5-brane solutions. Details of these applications will be given in [17].

Acknowledgements

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